Concordia University March 23, 2007

Engineering Differential Equations
Section J
Exam II (A)

ANSWER KEY

(1) (10 points) Solve the equation

$$xy' - y = 4xy^2, \quad x > 0.$$

Solution: $xy' - y = 4xy^2$ is a Bernoulli equation with $n = 2$. Denoting $v = y^{-1}$ and evaluating $v' = -y^{-2}y'$, the original equation becomes $v' + \frac{1}{x}v = -4$. This is a linear first-order differential equation which has an integrating factor of x. So $xv = -2x^2 + C$, where C is a constant, and thus $y(x) = \frac{x}{C - 2x^2}$, $C =$ arbitrary constant, is the general solution of the given Bernoulli equation.

□

(2) (10 points) A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Pure water is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at a rate of 5 L/min.

Solution: **(a)** Let $A(t)$ denote the number of grams of salt existent in the tank at time t. Then $A'(t) = R_{in} - R_{out} = 0 - R_{out}$ and therefore it satisfies the differential equation

$$\frac{dA}{dt} = -\frac{A}{200 - t} \cdot 5.$$

This is a linear, and a separable, differential equation. Using the resolution for separable equations, we obtain that its general solution is

$$\ln A(t) = 5 \ln |200 - t\,| + C \quad \Rightarrow \quad A(t) = K|200 - t\,|^5,$$

where C, respectively K, is an arbitrary constant.

Since $A(0) = 30$, we find that $K = 30/(200)^5$. Consequently

$$A(t) = \frac{3}{200^5} \cdot (200 - t)^5, \quad \text{thus} \quad A(5) = 30 \cdot \left(\frac{195}{200}\right)^5 \approx 26.43\,\text{g}.$$

(b) If $V(t)$ denotes the amount of water in the tank at time t, then $V(t)$ satisfies the differential equation $V'(t) = -1$. Hence $V(t) = -t + C$, where $C = V(0) = 200$. Consequently, $V(t) = 200 - t$ liters and $V(t) = 0 \ \Rightarrow \ t = 200$ minutes.

(c) The tank is half full at time $t = 100$. Therefore

$$A(100) = \frac{3}{200^5} \cdot (200 - 100)^5 = 30 \cdot \left(\frac{1}{2}\right)^5 = \frac{30}{32} \approx 0.94 \text{ grams}.$$

□

(3) **Solution:**

(a) "The maximum number of linearly independent solutions of an equation of the form $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$, with $a_i \in \mathbb{R},\ n \geq 2$, is n." is TRUE.

(b) "If $y_1(x) = \sin(x)$ is a solution of a homogeneous second-order differential equation with constant real coefficients, then $y_2(x) = \cos(x)$ is also a solution of this equation." is TRUE.

(c) The set of functions $f_1(x) = x,\ f_2(x) = \ln(x),\ f_3(x) = \ln(x^2)$ is linearly dependent on the interval $(0, \infty)$. One can see that $f_3(x) = 2f_2(x)$, hence f_2 and f_3 are linearly dependent, which implies that the set f_1, f_2, f_3 is linearly dependent too. This problem can also be solved using the Wronskian of the three functions:

$$W(f_1, f_2, f_3)(x) = \det \begin{pmatrix} x & \ln(x) & 2\ln(x) \\ 1 & \frac{1}{x} & \frac{2}{x} \\ 0 & -\frac{1}{x^2} & -\frac{2}{x^2} \end{pmatrix} = x \det \begin{pmatrix} \frac{1}{x} & \frac{2}{x} \\ -\frac{1}{x^2} & -\frac{2}{x^2} \end{pmatrix} - \det \begin{pmatrix} \ln(x) & 2\ln(x) \\ -\frac{1}{x^2} & -\frac{2}{x^2} \end{pmatrix} = 0,$$

for all $x > 0$. Thus, by Theorem 3.3, the functions f_1, f_2, f_3 are linearly dependent.

(d) To solve the initial value problem $y'' - y' = 0,\ y(0) = 1,\ y'(0) = -1$, consider first the auxiliary equation of the DE $y'' - y' = 0$. This is $m^2 - m = 0$ which has the real roots: $m_1 = 0$ and $m_2 = 1$. Therefore the general solution of the DE is $y_{gen}(x) = C_1 + C_2 e^x$, where C_1, C_2 are arbitrary constants. We will now use the initial conditions to find C_1 and C_2. We have that

$$y(0) = C_1 + C_2 = 1 \text{ and } y'(0) = C_2 = -1,$$

so $C_1 = 2$ and $C_2 = -1$. The solution to the given IVP is then

$$y(x) = 2 - e^x.$$

□

(4) (10 points) Find the general solution of the differential equation

$$y'' + 5y' + 4y = 24e^{-4x}.$$

Solution: Consider first the associated homogeneous equation $y'' + 5y' + 4y = 0$. The roots of its auxiliary equation $m^2 + 5m + 4 = 0$ are $m_1 = -1$ and $m_2 = -4$. Consequently,

$$y_c(x) = C_1 e^{-x} + C_2 e^{-4x},$$

where C_1 and C_2 are arbitrary constants.

To complete the problem, we need to find now a particular solution to the given non-homogeneous DE. To do this, we will use the method of undetermined coefficients. Given the form of $g(x) = 24e^{-4x}$, and that of $y_c(x)$ which overlaps with the form of $g(x)$, we assume that $y_p(x) = Axe^{-4x}$. Then

$$y_p'(x) = Ae^{-4x} - 4Axe^{-4x}, \quad y_p''(x) = -4Ae^{-4x} - 4Ae^{-4x} + 16Axe^{-4x},$$

so

$$-8Ae^{-4x} + 5Ae^{-4x} = 24Ae^{-4x} \quad \Rightarrow \quad A = -8.$$

Therefore $y_p(x) = -8xe^{-4x}$, and the general solution of the given DE is

$$y_{gen}(x) = C_1 e^{-x} + C_2 e^{-4x} - 8xe^{-4x},$$

where C_1 and C_2 are arbitrary constants.

□