Concordia University February 13, 2009

Applied Differential Equations
Section J
Exam I (A)

ANSWER KEY

(1) (10 points) The ODE

$$\frac{dy}{dx} = 4(y^2 + 1)$$

is separable. It can be written in the form

$$\frac{dy}{y^2 + 1} = 4\,dx,$$

which, by integration leads to the solution in implicit form

$$\arctan y = 4x + C,$$

where C is an arbitrary constant.

$\square$

(2) (10 points) The following equation

$$\frac{dy}{dx} + y = xy^4$$

is a Bernoulli equation with $n = 4$. We will use the substitution $u = y^{-3}$ (with $du = -3y^{-4}\,dy$).

Following this substitution, we obtain the linear ODE

$$\frac{du}{dx} - 3u = -3x.$$

An integrating factor is then $\mu(x) = e^{-3x}$, so $e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3e^{-3x}x$, or

$$(e^{-3x}u)' = -3xe^{-3x}.$$

Consequently, $e^{-3x}u = -3\int xe^{-3x}\,dx + C$, or (using integration by parts) $e^{-3x}u = xe^{-3x} - \int e^{-3x}\,dx + C$.

Finally, $e^{-3x}u = xe^{-3x} + \frac{1}{3}e^{-3x} + C$, so $u = x + \frac{1}{3} + Ce^{3x}$ and $y = (x + \frac{1}{3} + Ce^{3x})^{-1/3}$, where C is an arbitrary constant.

$\square$

(3) Given

$$\frac{x}{2y^4}\,dx + \left(\frac{3y^2 - x^2}{y^5} + \sqrt{2y}\right)\,dy = 0,$$

we consider $f(x, y)$ such that

$$\frac{\partial f}{\partial x}(x, y) = \frac{x}{2y^4}$$
$$\frac{\partial f}{\partial y}(x, y) = \frac{3y^2 - x^2}{y^5} + \sqrt{2y}.$$

Integrating the first equation with respect to x, we obtain

$$f(x, y) = \frac{x^2}{4y^4} + c(y).$$

To find $c(y)$, we evaluate

$$\frac{\partial f}{\partial y}(x, y) = -\frac{x^2}{y^5} + c'(y) = \frac{3y^2 - x^2}{y^5} + \sqrt{2y},$$

hence $c(y)$ satisfies the ODE: $c'(y) = \frac{3}{y^3} + \sqrt{2y}$. Therefore $c(y) = -\frac{3}{2y^2} + \frac{(2y)^{3/2}}{3} + C$, where C is an arbitrary constant.

Finally we have that $f(x, y) = \frac{x^2}{4y^4} - \frac{3}{2y^2} + \frac{(2y)^{3/2}}{3} + C$, and we conclude that the general solution of the given exact equation is (in implicit form)

$$\frac{x^2}{4y^4} - \frac{3}{2y^2} + \frac{(2y)^{3/2}}{3} = c, \quad \text{where } c \text{ is an arbitrary constant.}$$

Now, $y(0) = 2$ implies $c = 55/24$, hence the solution of the IVP

$$\frac{x^2}{4y^4} - \frac{3}{2y^2} + \frac{(2y)^{3/2}}{3} = \frac{55}{24}.$$

□

(4) Denote by $A(t)$ the number of pounds of salt in the tank at time t. Then we must solve the IVP

$$\frac{dA}{dt} = 12 - \frac{5A}{100 - t}, \quad A(0) = 0.$$

The ODE is linear with an integrating factor $\mu(t) = \exp\left(\int \frac{5}{100 - t}\,dt\right) = \exp(-5\ln(100 - t)) = (100 - t)^{-5}$.

So, $(100-t)^{-5}A'(t) + 5A(100-t)^{-4} = 12(100-t)^{-5}$, hence $(100-t)^{-5}A(t) = \frac{12}{4}(100-t)^{-4} + C$ where C will be determined from $A(0) = 0$ to be equal to $-3/100^4$.

Finally, we have $A(t) = 3(100 - t) - \frac{3}{100^4}(100 - t)^5$ and $A(30) = 210 - \frac{21}{10^3} \approx 209.97$ lb.

□