Concordia
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ENGINEERING MATHEMATICS 212 - SUPPLEMENTARY NOTES

Quadric Surfaces

A surface whose equation is $F(x, y, z) = 0$, where $F(x, y, z)$ has the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$$

(i.e. $F(x, y, z)$ is a general second degree polynomial in x, y and z) is called a quadric surface. The purpose of these notes is to classify the more frequently encountered quadric surfaces which can be obtained from the above general formula and to provide some clues for sketching and visualizing these surfaces.

The situation is quite similar to the problem of classifying second degree curves in the $x - y$ plane of the form

$$Ax^2 + By^2 + Dxy + Gx + Hy + J = 0$$

In this case, we know that by an appropriate rotation and/or translation of axes, the equation transforms to the equation of an ellipse, hyperbola, or parabola (or a degenerate conic, such as one or two straight lines) in standard form.

In the case of quadric surfaces, we proceed as follows:

First - it is always possible to eliminate the terms $Dxy + Eyz + Fxz$ (i.e. the "cross terms") by a suitable 3 dimensional rotation of the coordinate system (this will be shown in EMAT 232).

Second - by a suitable translation of the new axes, it is possible to eliminate those linear terms (i.e. Gx, Hy, Iz) for which the corresponding quadratic term (respectively: Ax^2, By^2, Cz^2) is not identically zero. For example if $A \neq 0$, which means Ax^2 is not identically zero, we can eliminate the term Gx (by completing squares on $Ax^2 + Gx$, which leads to a translation of the x-axis).

Third - in an equation such as $Ax^2 + By^2 + Iz + J = 0$, i.e. an equation where a non-zero linear term actually appears, we can eliminate the constant term J by translating along the z-axis. Simply put $z' = z + \frac{J}{I}$. The transformed equation now reads $Ax^2 + By^2 + Iz' = 0$.

Fourth - an equation of the form $Ax^2 + Hy + Iz = 0$ (two linear terms) can be put in the form $Ax^2 + I'z' = 0$ (a single linear term) by a rotation in the $y - z$ plane: simply rotate until the z-axis points along the line $Hy + Iz = 0$ in the $y - z$ plane, and call the rotated axis the $z' - axis$.

NOTE: The equation of a particular surface depends on its orientation with respect to the coordinate axis; e.g. $z = x^2 + y^2$ and $y = x^2 + z^2$ are the same surface, the latter opening to the right, the former opening up.

Finally, - by dividing each equation by an appropriate non-zero constant and a minor change of notation, we have the following basic types of equations to consider:

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} \pm 1 = 0 \qquad \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 0$$

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0 \qquad \pm\frac{x^2}{a^2} \pm 2z = 0$$

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm 2z = 0 \qquad \pm\frac{x^2}{a^2} \pm 1 = 0$$

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm 1 = 0 \qquad \pm\frac{x^2}{a^2} = 0$$

NOTE: whether $x^2 + y^2 = 4$ is the equation of a circle in the xy plane or that of a cylinder can be only deduced from the context.

It turns out that these equations represent a total of 17 different types of quadric surfaces in standard position. Of these, 14 are real and 3 imaginary (in the sense that the equations are not satisfied by any real numbers (x, y, z), so that the sets of points of the corresponding surfaces are empty).

We will look in detail at the quadric surfaces that will be met in this course. The most frequently encountered have the property that they are surfaces of revolution: they are generated by rotating a curve about an axis, and the intersection of the surface with a plane perpendicular to this axis is a circle. Rotating the curve $\begin{cases} z = x^2 \\ y = 0 \end{cases}$ about the z axis yields the paraboloid $z = x^2 + y^2$.

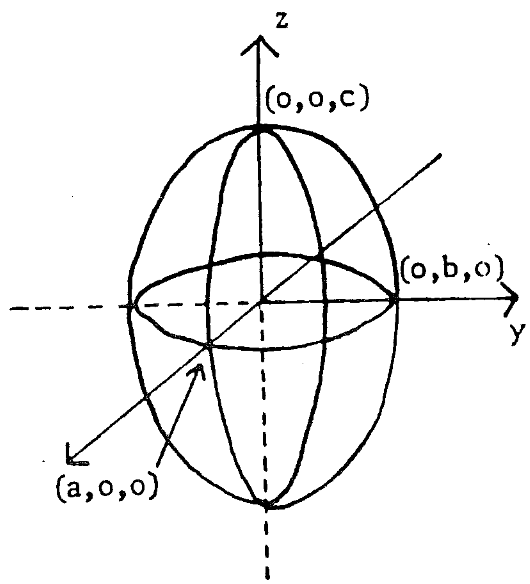
1. ELLIPSOID: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

To assist our visualization of a surface we consider its curves of intersection with various planes, most commonly planes which are parallel to the coordinate planes. Such curves are called traces of the surface (traces on planes parallel to the $x - y$ plane are also called level curves). In particular, the traces on the coordinate planes themselves are usually the most instructive, and we will denote them respectively by: T_{xy} ($x - y$ plane); T_{yz} ($y - z$ plane); T_{xz} ($x - z$ plane). Algebraically one obtains the trace, say on the $z = k$ plane, by simply substituting $z = k$ into the equation, and interpreting the resulting 2-variable equation as that of a curve lying in the plane $z = k$. Letting $z = 0$ gives T_{xy}. In this case we have:

$x - y$ trace $(z = k)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$

(ellipse if $\frac{k^2}{c^2} < 1$, a point if $k^2 = c^2$, and no intersection if $k^2 > c^2$.)

$T_{xy} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. And similarly for T_{xz} trace $(y = 0)$: $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$



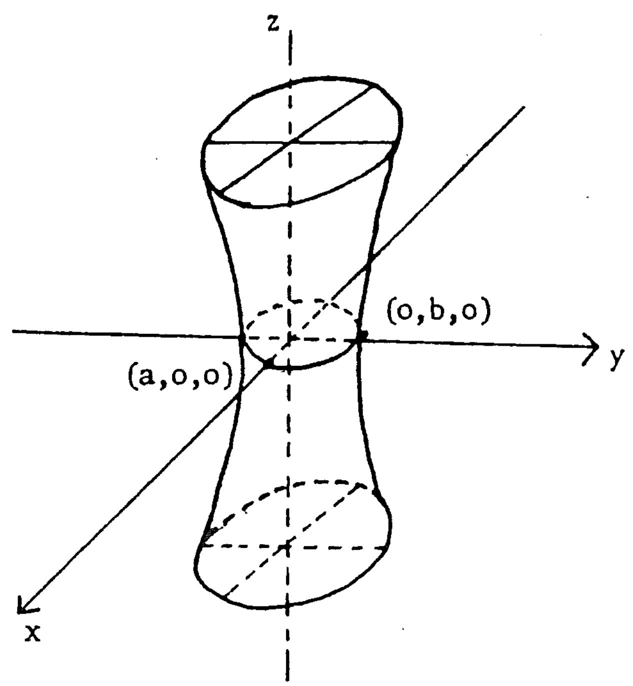
If $a^2 = b^2 = c^2$ the ellipsoid is actually a sphere of radius $|a|$.

2. HYPERBOLOID OF ONE SHEET (Simple Case): $x^2+y^2-z^2-1 = 0$

This surface is generated by rotating the hyperbola $\begin{cases} y^2 - z^2 = 1 \\ x = 0 \end{cases}$

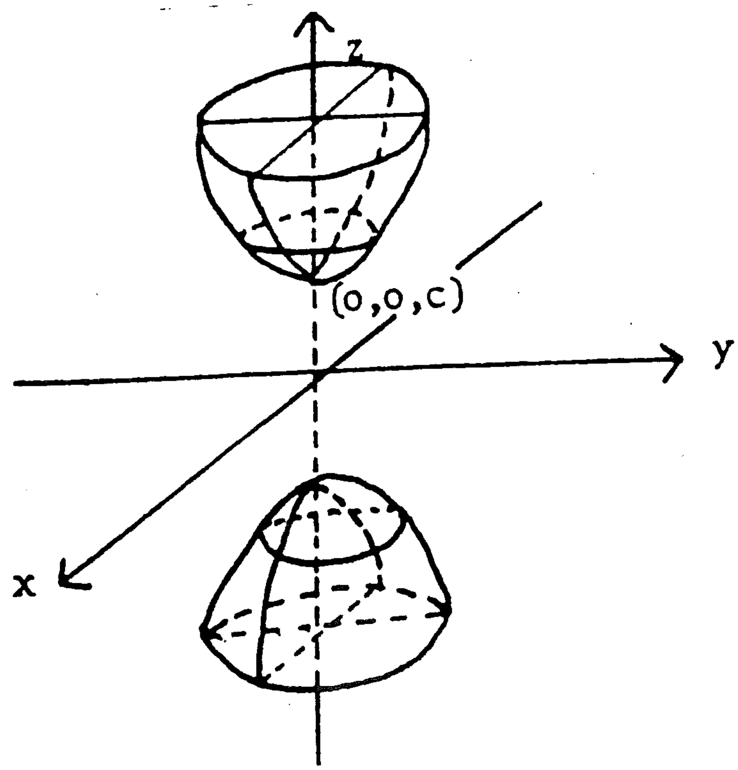
about the z axis.

HYPERBOLOID OF ONE SHEET



Writing as $x^2 + y^2 = 1 + z^2$, we see that the right hand side ≥ 1, so there are no points closer than one unit to the z axis.

3. HYPERBOLOID OF TWO SHEETS $z^2 - x^2 - y^2 = 1$



Note that $z^2 = 1 + x^2 + y^2 \geq 1$ so there are no points with $|z| < 1$.

4. ELLIPTIC CONE: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

If $a = b$ then we have a circular cone

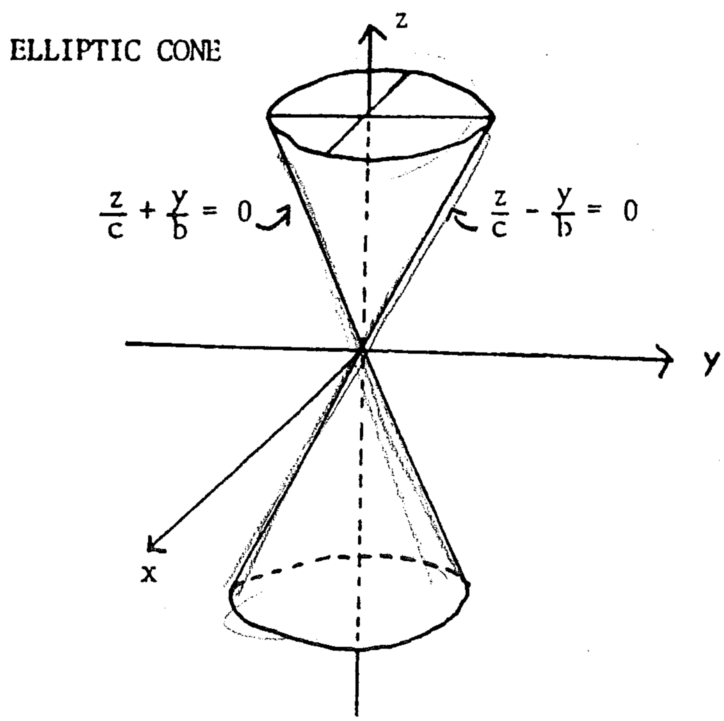
$x - y$ trace: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{k^2}{c^2}$ (ellipse, circle if $a = b$)

$x - z$ trace: $-\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = \dfrac{k^2}{b^2}$ (hyperbola if $k \neq 0$)

$Txz : \left(\dfrac{z}{c} - \dfrac{x}{a}\right)\left(\dfrac{z}{c} + \dfrac{x}{a}\right) = 0$ (two straight lines)

$y - z$ trace: $-\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = \dfrac{k^2}{a^2}$ (hyperbola if $k \neq 0$)

$Tyz : \left(\dfrac{z}{c} - \dfrac{y}{b}\right)\left(\dfrac{z}{c} + \dfrac{y}{b}\right) = 0$ (two straight lines)



For $z = \sqrt{3}\,\sqrt{x^2 + y^2}$ angle α above is $\dfrac{\pi}{6}$: trace in $y - z$ plane is $z = \sqrt{3}\,|y|$, so angle between y axis and line $z = \sqrt{3}\,|y|$ is $\dfrac{\pi}{3}$, so $\alpha = \dfrac{\pi}{2} - \dfrac{\pi}{3} = \dfrac{\pi}{6}$.

5. ELLIPTIC PARABOLOID: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$

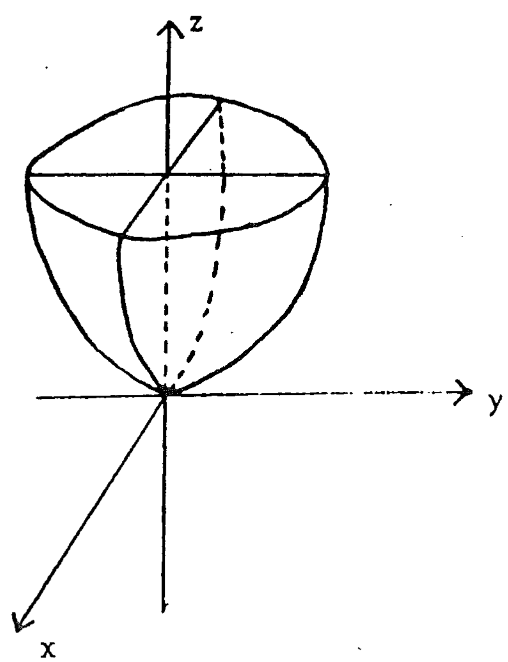
$x - y$ trace: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = K$ (ellipse if $K > 0$ and a circle if $a = b$)

Txz: is the parabola $z = \frac{x^2}{a^2}$

Tyz: parabola $z = \frac{y^2}{b^2}$

Note that $z = 4 - x^2 - y^2$ is the equation of a paraboloid opening down from $(0,0,4)$.

ELLIPTIC PARABOLOID



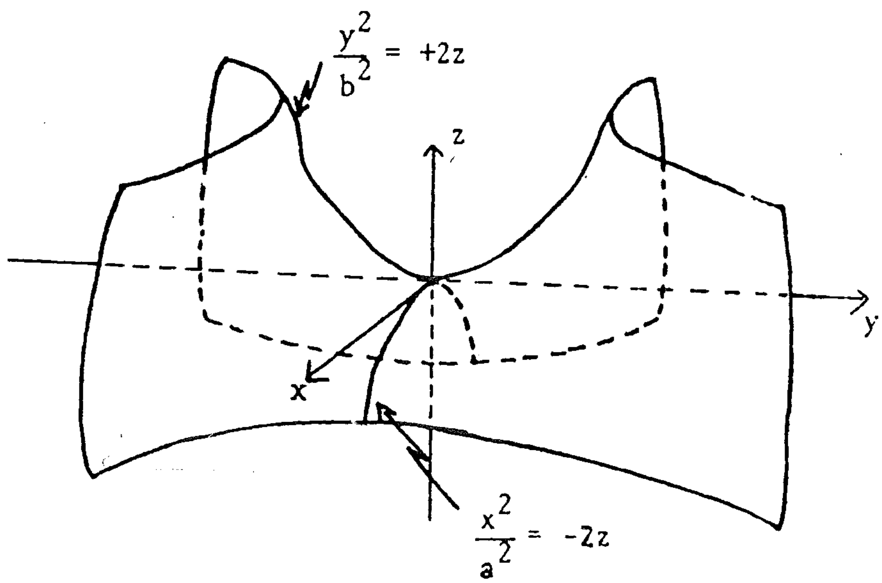
6. HYPERBOLIC PARABOLOID: $\frac{x^2}{a^2} - \frac{y^2}{b^2} + z = 0$

$x-y$ trace: $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=K$ (hyperbola if $K \neq 0$, two straight linges if $K=0$)

$x-z$ trace: $\frac{x^2}{a^2} = -2\left(z-\frac{K^2}{2b^2}\right)$ (parabola)

$y-z$ trace: $\frac{y^2}{b^2} = -2\left(z+\frac{K^2}{2a^2}\right)$ (parabola)

HYPERBOLIC PARABOLOID



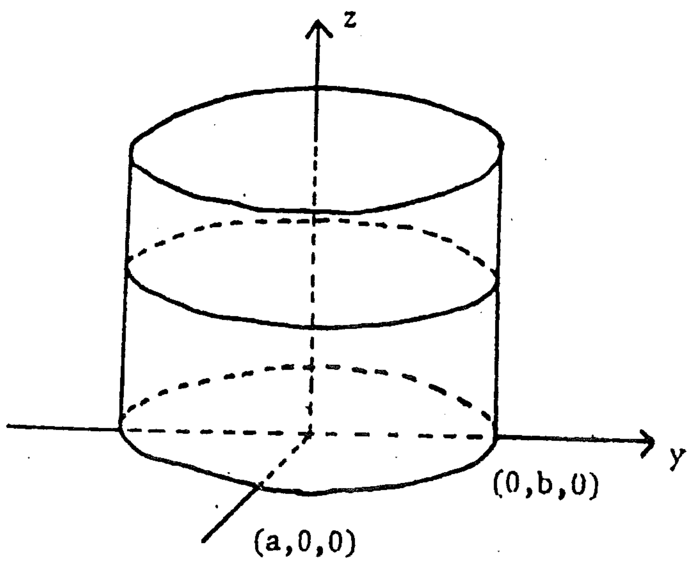
This surface is the graph of a function which can have both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at a point which is neither a minimum nor a mximum for $f(x,y)$.

7. ELLIPTIC CYLINDER: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$x - y$ trace: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

independent of z (ellipse). This surface consists of all lines parallel to the z-axis and passing through the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the $x - y$ plane.

ELLIPTIC CYLINDER



Special Case: circular cylinder $x^2 + y^2 = a^2$. All points are at a distance $|a|$ from the z axis.

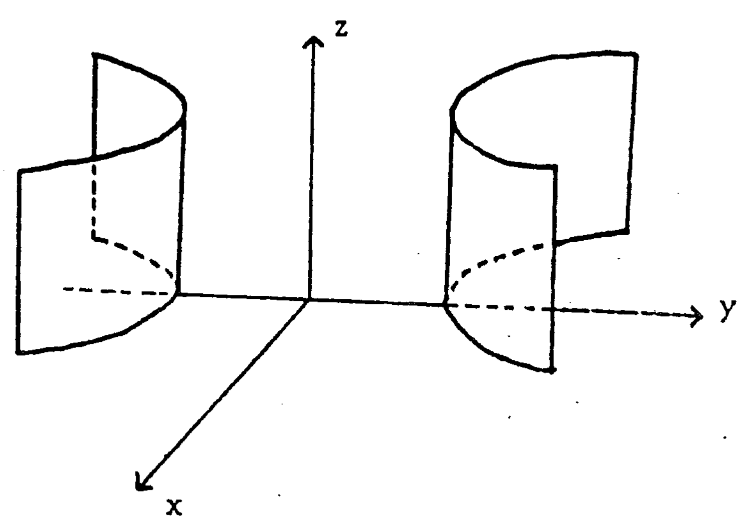
Same surface, but oriented differently: $x^2 + z^2 = a^2, \quad y^2 + z^2 = a^2$.

8. HYPERBOLIC CYLINDER: $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$x - y$ trace: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

independent of z (hyperbola). Thus the surface consists of all lines parallel to the z-axis and passing through the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

HYPERBOLIC CYLINDER



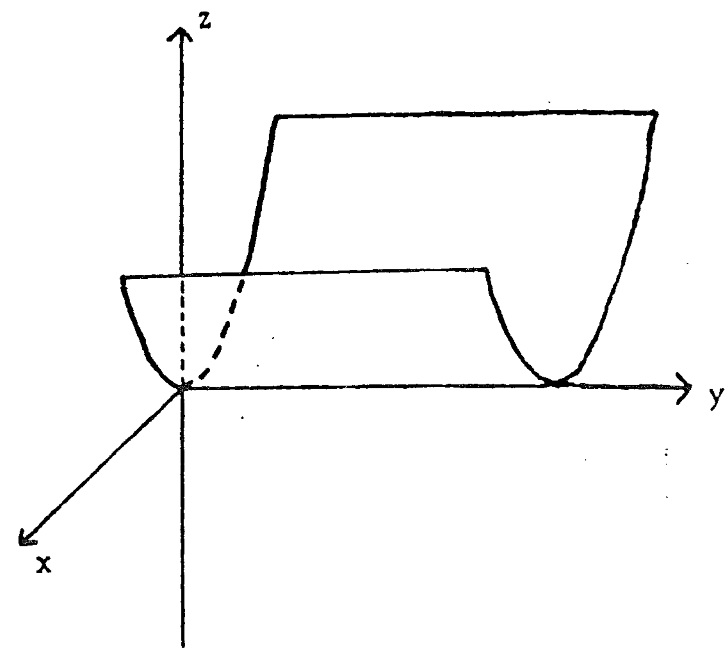
9. PARABOLIC CYLINDER: $\frac{x^2}{a^2} - z = 0$

$y - z$ trace: $z = \frac{K^2}{a^2}$ (lines parallel to y-axis)

$x - z$ trace: $x^2 = a^2 z$ (independent of y – parabola)

Thus the surface consists of all lines parallel to the y-axis and passing through the parabola $\frac{x^2}{a^2} - z = 0$

PARABOLIC CYLINDER



Week 7

Name and sketch the following quadric surfaces:

1. $36x^2 + 9y^2 + 4z^2 - 36 = 0$
2. $z = 1 - \sqrt{x^2 + y^2}$
3. $y^2 + z^2 = x$
4. $7x^2 - 3y^2 + z = 0$
5. $x^2 - y^2 = 1$
6. $3x^2 + 3y^2 - z^2 = 0$
7. $4x^2 + 9y^2 = 36$
8. $x^2 + y^2 + z^2 = 8z$
9. $z = 9 - x^2 - y^2$
10. Find the curve of intersection of the surfaces $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.
11. Find the curve of intersection of the surfaces $x^2 + y^2 + z^2 = 20$ and $z = x^2 + y^2$.

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